Chapter 4 Determinants

EXERCISE 4.1

Question 1:

Evaluate the determinant $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

Let $|A| = \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Hence,

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} \\ &= 2(-1) - 4(-5) \\ &= -2 + 20 \\ &= 18 \end{aligned}$$

Question 2:

Evaluate the determinants:

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

(i)

$$\begin{aligned} \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} &= (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

(ii)

$$\begin{aligned} \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} &= (x^2 - x + 1)(x + 1) - (x - 1)(x + 1) \\ &= x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1) \\ &= x^3 + 1 - x^2 + 1 \\ &= x^3 - x^2 + 2 \end{aligned}$$

Question 3:

If $A=\begin{bmatrix}1 & 2\\4 & 2\end{bmatrix}$, then show that $|2A|=4|A|$

Solution:

The given matrix is $A=\begin{bmatrix}1 & 2\\4 & 2\end{bmatrix}$

Therefore,

$$2A=2\begin{pmatrix}1 & 2\\4 & 2\end{pmatrix}$$
$$=\begin{pmatrix}2 & 4\\8 & 4\end{pmatrix}$$

Hence,

$$LHS=|2A|$$
$$=\begin{vmatrix}2 & 4\\8 & 4\end{vmatrix}$$
$$=2\times4-4\times8$$
$$=8-32$$
$$=-24$$

Now,

$$|A|=\begin{vmatrix}1 & 2\\4 & 2\end{vmatrix}=1\times2-2\times4$$
$$=2-8$$
$$=-6$$

Therefore,

$$RHS=4|A|$$
$$=4(-6)$$
$$=-24$$

Thus, $|2A|=4|A|$ proved.

Question 4:

If $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{pmatrix}$, then show that $|3A|=27|A|$

Solution:

The given matrix is $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{pmatrix}$

It can be observed that in the first column, two entries are zero. Thus, we expand along the first column (C_1) for easier calculation.

$$\begin{aligned}|A| &= 1\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix}-0\begin{vmatrix}0 & 1\\ 1 & 4\end{vmatrix}+0\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}\\ &= 1(4-0)-0+0\\ &= 4\end{aligned}$$

Therefore,

$$\begin{aligned}27|A| &= 27|4|\\ &= 108 \qquad \ldots(1)\end{aligned}$$

Now,

$$3A=3\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{pmatrix}=\begin{pmatrix}3 & 0 & 3\\ 0 & 3 & 6\\ 0 & 0 & 12\end{pmatrix}$$

Therefore,

$$\begin{aligned}|3A| &= 3\begin{vmatrix}3 & 6\\ 0 & 12\end{vmatrix}-0\begin{vmatrix}0 & 3\\ 0 & 12\end{vmatrix}+0\begin{vmatrix}0 & 3\\ 3 & 6\end{vmatrix}\\ &= 3(36-0)\\ &= 36(36)\\ &= 108 \qquad \ldots(2)\end{aligned}$$

From equations (1) and (2),

$$|3A|=27|A|$$

Thus, $|3A|=27|A|$ proved.

Question 5:

Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

(ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

(iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Solution:

(i) Let $A = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

It can be observed that in the second row, two entries are zero. Thus, we expand along the second row for easier calculation.

Hence,

$$|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix}$$

$$= (-15+3)$$

$$= -12$$

(ii) Let $A = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Hence,

$$|A| = 3\begin{vmatrix}1 & -2\\3 & 1\end{vmatrix} + 4\begin{vmatrix}1 & -2\\2 & 1\end{vmatrix} + 5\begin{vmatrix}1 & 1\\2 & 3\end{vmatrix}$$
$$= 3(1+6)+4(1+4)+5(3-2)$$
$$= 3(7)+4(5)+5(1)$$
$$= 21+20+5$$
$$= 46$$

(iii) Let $A = \begin{vmatrix}0 & 1 & 2\\-1 & 0 & -3\\-2 & 3 & 0\end{vmatrix}$

Hence,

$$|A| = 0\begin{vmatrix}0 & -3\\3 & 0\end{vmatrix} - 1\begin{vmatrix}-1 & -3\\-2 & 0\end{vmatrix} + 2\begin{vmatrix}-1 & 0\\-2 & 3\end{vmatrix}$$
$$= 0-1(0-6)+2(-3-0)$$
$$= -1(-6)+2(-3)$$
$$= 6-6=0$$

(iv) Let $A = \begin{vmatrix}2 & -1 & -2\\0 & 2 & -1\\3 & -5 & 0\end{vmatrix}$

Hence,

$$|A| = 2\begin{vmatrix}2 & -1\\-5 & 0\end{vmatrix} - 0\begin{vmatrix}-1 & -2\\-5 & 0\end{vmatrix} + 3\begin{vmatrix}-1 & -2\\2 & -1\end{vmatrix}$$
$$= 2(0-5)-0+3(1+4)$$
$$= -10+15=5$$

Question 6:

If $A = \begin{pmatrix}1 & 1 & -2\\2 & 1 & -3\\5 & 4 & -9\end{pmatrix}$, find $|A|$

Solution:

Let $A = \begin{pmatrix}1 & 1 & -2\\2 & 1 & -3\\5 & 4 & -9\end{pmatrix}$

Hence,

$$|A| = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$
$$= 1(-9+12) - 1(-18+15) - 2(8-5)$$
$$= 1(3) - 1(-3) - 2(3)$$
$$= 3 + 3 - 6$$
$$= 0$$

Question 7:

Find the values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Therefore,

$$\Rightarrow 2 \times 1 - 5 \times 4 = 2x \times x - 6 \times 4$$
$$\Rightarrow 2 - 20 = 2x^2 - 24$$
$$\Rightarrow 2x^2 = 6$$
$$\Rightarrow x^2 = 3$$
$$\Rightarrow x = \pm\sqrt{3}$$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Therefore,

$$\Rightarrow 2 \times 5 - 3 \times 4 = x \times 5 - 3 \times 2x$$
$$\Rightarrow 10 - 12 = 5x - 6x$$
$$\Rightarrow -2 = -x$$
$$\Rightarrow x = 2$$

Question 8:

If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, the x is equal to

(A) 6 (B) ±6 (C) −6 (D) 0

Solution:

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

Therefore,

$$\Rightarrow x^2 - 36 = 36 - 36$$
$$\Rightarrow x^2 - 36 = 0$$
$$\Rightarrow x^2 = 36$$
$$\Rightarrow x = \pm 6$$

Thus, the correct option is B.

EXERCISE 4.2

Question 1:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Solution:

$$\Delta = \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$$

$$= \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix}$$

Here, two columns of each determinant are identical.

Hence,

$$\begin{aligned} \Delta &= 0+0 \\ &= 0 \end{aligned}$$

Question 2:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Solution:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix} \qquad [R_1 \rightarrow R_1 + R_2]$$

$$= \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

Hence, $\Delta = 0$

Question 3:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Solution:

$$\Delta = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix} + 0 \qquad [\because \text{Two columns are identical}]$$

$$= 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

$$= 0 \qquad [\because \text{Two columns are identical}]$$

Question 4:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution:

$$\Delta=\begin{vmatrix}1 & bc & a(b+c)\\ 1 & ca & b(c+a)\\ 1 & ab & c(a+b)\end{vmatrix}$$

$$=\begin{vmatrix}1 & bc & ab+bc+ca\\ 1 & ca & ab+bc+ca\\ 1 & ab & ab+bc+ca\end{vmatrix} \qquad [C_3 \to C_3 + C_2]$$

Here, the two columns C_1 and C_3 are proportional.
Hence, $\Delta = 0$

Question 5:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}=2\begin{vmatrix}a & p & x\\ b & q & y\\ c & r & z\end{vmatrix}$$

Solution:

$$\Delta=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}$$

$$=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a & p & x\end{vmatrix}+\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ b & q & y\end{vmatrix}$$

$$=\Delta_1+\Delta_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now,

$$\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

$$= \begin{vmatrix} b+c & q+r & y+z \\ c & r & z \\ a & p & x \end{vmatrix} \qquad [R_2 \to R_2 - R_3]$$

$$= \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix} \qquad [R_1 \to R_1 - R_2]$$

$$= (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad [R_1 \leftrightarrow R_3 \text{ and } R_2 \leftrightarrow R_3] \qquad \ldots(2)$$

$$\Delta_2 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix} \qquad [R_1 \to R_1 - R_3]$$

$$\Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix} \qquad [R_2 \to R_2 - R_1]$$

$$\Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$= \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad [R_1 \leftrightarrow R_2 \textit{ and } R_2 \leftrightarrow R_3] \qquad \ldots(3)$$

From $(1),(2)$ and (3), we have

$$\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} + \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Hence, $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ proved.

Question 6:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Solution:

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$= \frac{1}{c}\begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \qquad [R_1 \to cR_1]$$

$$= \frac{1}{c}\begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \qquad [R_1 \to R_1 - bR_2]$$

$$= \frac{a}{c}\begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

Hence, $\Delta = 0$

Question 7:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \qquad [\text{Taking out factors } a, b, c \text{ from } R_1, R_2, R_3]$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad [\text{Taking out factors } a, b, c \text{ from } C_1, C_2, C_3]$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} \qquad [R_2 \to R_2 + R_1 \text{ and } R_3 \to R_3 + R_1]$$

$$= a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -a^2b^2c^2(0-4)$$

$$= 4a^2b^2c^2$$

Question 8:

By using properties of determinants show that:

(i) $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Solution:

(i) Let $$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \qquad [R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & -1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} \qquad [R_1 \to R_1 + R_2]$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} 0 & -1 \\ 1 & b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

Hence, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ proved.

(ii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix} \qquad [C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$,

$$\Delta=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(b-c)(c-a)(a-b)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -(a+b+c) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -1 & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix}0 & 1\\ 1 & c\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)$$

Hence, $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$ proved.

Question 9:

By using properties of determinants show that:

$$\begin{vmatrix}x & x^2 & yz\\ y & y^2 & zx\\ z & z^2 & xy\end{vmatrix}=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution:

Let $\Delta=\begin{vmatrix}x & x^2 & yz\\ y & y^2 & zx\\ z & z^2 & xy\end{vmatrix}$

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= \begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix}$$

$$= (x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & (z+x) & -y \end{vmatrix}$$

$$\Delta = (x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & z-y & z-y \end{vmatrix} \qquad [R_3 \to R_3 + R_2]$$

$$= (x-y)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix} x & yz \\ -1 & z \end{vmatrix} + 1\begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix}\right]$$

$$= (x-y)(z-x)(z-y)\left[(-xz-yz)+(-x^2-xy+x^2)\right]$$

$$= -(x-y)(z-x)(z-y)(xy+yz+zx)$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence, $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$ proved.

Question 10:

By using properties of determinants show that:

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Solution:

(i) $$\Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix} \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= (5x+4)(4-x)(4-x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2\begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

Hence, $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$ proved.

(ii) $$\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= (3y+k)\begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= (3y+k)\begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix} \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= k^2(3y+k)\begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding along C_3

$$\Delta = k^2(3y+k)\begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix}$$

$$= k^2(3y+k)$$

Hence, $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$ proved.

Question 11:

By using properties of determinants show that:

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Solution:

(i) $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= (a+b+c)^3\begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & -1 \end{vmatrix}$$

$$= (a+b+c)^3(-1)(-1)$$

$$= (a+b+c)^3$$

Hence, proved.

(ii)

$$\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= 2(x+y+z)^3\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(x+y+z)^3(1)(1-0)$$

$$= 2(x+y+z)^3$$

Hence, proved.

Question 12:

By using properties of determinants show that:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \left(1-x^3\right)^2$$

Solution:

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= \left(1+x+x^2\right)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$\Delta = \left(1+x+x^2\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix} \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$\Delta = \left(1+x+x^2\right)(1-x)(1-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)(1-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

Expanding along R_1

$$\Delta = \left(1-x^3\right)(1-x)(1)\begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)(1-x)\left(1+x+x^2\right)$$

$$= \left(1-x^3\right)\left(1-x^3\right)$$

$$= \left(1-x^3\right)^2$$

Hence, proved.

Question 13:

By using properties of determinants show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Solution:

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right) \\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \qquad \left[R_1 \to R_1 + bR_3 \text{ and } R_2 \to R_2 - aR_3\right]$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \left[(1)\begin{vmatrix} 1 & a \\ -2a & 1-a^2-b^2 \end{vmatrix} - b\begin{vmatrix} 0 & 1 \\ 2b & -2a \end{vmatrix}\right]$$

$$= \left(1+a^2+b^2\right)^2 \left[1-a^2-b^2+2a^2-b(-2b)\right]$$

$$= \left(1+a^2+b^2\right)^2 \left(1+a^2+b^2\right)$$

$$= \left(1+a^2+b^2\right)^3$$

Hence, proved.

Question 14:

By using properties of determinants show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution:

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors a,b,c from R_1, R_2, R_3 respectively,

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= abc \times \frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \qquad [C_1 \to aC_1, C_2 \to bC_2 \text{ and } C_3 \to cC_3]$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1\left(-c^2\right) + \left(a^2 + 1 + b^2\right)$$

$$= 1 + a^2 + b^2 + c^2$$

Hence, proved.

Question 15:

Let A be a square matrix of order 3×3, then $|kA|$ is equal to:

(A) $k|A|$ (B) $k^2|A|$ (C) $k^3|A|$ (D) $3k|A|$

Solution:

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Then,

$$kA = \begin{pmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{pmatrix}$$

$$|kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

Taking out common factors k from each row

$$|kA| = k^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k^3 |A|$$

The correct option is C.

Question 16:

Which of the following is correct?

(A) Determinant is a square matrix.
(B) Determinant is a number associated to a matrix.
(C) Determinant is a number associated to a square matrix.
(D) None of the above.

Solution:

We know that to every square matrix, $A = [a_{ij}]$ of order n, we can associate a number called the determinant of square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

Thus, the determinant is a number associated to a square matrix.

Hence, the correct option is C.

EXERCISE 4.3

Question 1:

Find area of the triangle with vertices at the point given in each of the following:

(i) $(1,0),(6,0),(4,3)$

(ii) $(2,7),(1,1),(10,8)$

(iii) $(-2,-3),(3,2),(-1,-8)$

Solution:

(i) The area of the triangle with vertices $(1,0),(6,0),(4,3)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[1(0-3)-0(6-4)+1(18-0)\right]$$

$$=\frac{1}{2}[-3+18]$$

$$=\frac{1}{2}[15]$$

$$=\frac{15}{2}$$

Hence, area of the triangle is $\frac{15}{2}$ square units.

(ii) The area of the triangle with vertices $(2,7),(1,1),(10,8)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}2 & 7 & 1\\ 1 & 1 & 1\\ 10 & 8 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$$

$$=\frac{1}{2}\left[2(-7)-7(-9)+1(-2)\right]$$

$$=\frac{1}{2}[-14+63-2]$$

$$=\frac{1}{2}[47]$$

$$=\frac{47}{2}$$

Hence, area of the triangle is $\frac{47}{2}$ square units.

(iii) The area of the triangle with vertices $(-2,-3),(3,2),(-1,-8)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]$$

$$= \frac{1}{2}\left[-2(10)+3(4)+1(-22)\right]$$

$$= \frac{1}{2}[-20+12-22]$$

$$= -\frac{1}{2}[30]$$

$$= -15$$

Hence, area of the triangle is 15 square units.

Question 2:

Show that the points $A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear.

Solution:

The area of the triangle with vertices $A(a,b+c),B(b,c+a),C(c,a+b)$ is given by the absolute value of the relation:

$$\Delta=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad [R_3 \to R_3 + R_2]$$

$$=0$$

Thus, the area of the triangle formed by points is zero.

Hence, the points are collinear.

Question 3:

Find values of k if area of triangle is 4 square units and vertices are:

(i) $(k,0),(4,0),(0,2)$

(ii) $(-2,0),(0,4),(0,k)$

Solution:

We know that the area of a triangle whose vertices are $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is the absolute value of the determinant (Δ), where

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It is given that the area of triangle is 4 square units.

Hence, $\Delta=\pm4$

(i) The area of the triangle with vertices $(k,0),(4,0),(0,2)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[k(0-2) - 0(4-0) + 1(8-0)\right]$$

$$= \frac{1}{2}\left[-2k+8\right]$$

$$= -k+4$$

Therefore, $-k+4 = \pm 4$

When $-k+4=-4$
Then $k=8$

When $-k+4=4$
Then $k=0$

Hence, $k=0,8$

(ii) The area of the triangle with vertices $(-2,0),(0,4),(0,k)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(4-k)\right]$$

$$= k-4$$

Therefore, $-k+4 = \pm 4$

When $k-4=4$
Then $k=8$

When $k-4=-4$
Then $k=0$

Hence, $k=0,8$

Question 4:

(i) Find equation of line joining $(1,2)$ and $(3,6)$ using determinants.

(ii) Find equation of line joining $(3,1)$ and $(9,3)$ using determinants.

Solution:

(i) Let $P(x,y)$ be any point on the line joining points $A(1,2)$ and $B(3,6)$.
Then, the points A,B and P are collinear.
Hence, the area of triangle ABP will be zero.
Therefore,

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[1(6-y)-2(3-x)+1(3y-6x)\right]=0$$

$$\Rightarrow 6-y-6+2x+3y-6x=0$$

$$\Rightarrow 2y-4x=0$$

$$\Rightarrow y=2x$$

Thus, the equation of the line joining the given points is $y=2x$.

(ii) Let $P(x,y)$ be any point on the line joining points $A(3,1)$ and $B(9,3)$.
Then, the points A,B and P are collinear.
Hence, the area of triangle ABP will be zero.
Therefore,

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$

$$\Rightarrow 9-3y-9+x+9y-3x=0$$

$$\Rightarrow 6y-2x=0$$

$$\Rightarrow x-3y=0$$

Thus, the equation of the line joining the given points is $x-3y=0$.

Question 5:

If area of the triangle is 35 square units with vertices $(2,-6),(5,4),(k,4)$. Then k is

(A) 12 (B) -2 (C) $-12,-2$ (D) $12,-2$

Solution:

The area of the triangle with vertices $(2,-6),(5,4),(k,4)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$= \frac{1}{2}[30-6k+20-4k]$$

$$= \frac{1}{2}[50-10k]$$

$$= 25-5k$$

It is given that the area of the triangle is 35 square units
Hence, $\Delta = \pm 35$.

Therefore,

$$\Rightarrow 25-5k = \pm 35$$

$$\Rightarrow 5(5-k) = \pm 35$$

$$\Rightarrow 5-k = \pm 7$$

When, $5-k=-7$
Then, $k=12$

When, $5-k=7$
Then, $k=-2$

Hence, $k=12,-2$

Thus, the correct option is D.

EXERCISE 4.4

Question 1:

Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution:

(i) The given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$M_{11} =$ minor of element $a_{11} = 3$

$M_{12} =$ minor of element $a_{12} = 0$

$M_{21} =$ minor of element $a_{21} = -4$

$M_{22} =$ minor of element $a_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) The given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$M_{11} =$ minor of element $a_{11} = d$

$M_{12} =$ minor of element $a_{12} = b$

$M_{21} =$ minor of element $a_{21} = c$

$M_{22} =$ minor of element $a_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$$
$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$$
$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$$
$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$$

Question 2:

Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Solution:

(i) The given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$M_{11} =$ minor of element $a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{12} =$ minor of element $a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$M_{13} =$ minor of element $a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$M_{21} =$ minor of element $a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$M_{22} =$ minor of element $a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{23} =$ minor of element $a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$M_{31} =$ minor of element $a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$M_{32} =$ minor of element $a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$M_{33} =$ minor of element $a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (1) = 1$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (0) = 0$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 (0) = 0$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 (0) = 0$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 (0) = 0$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 (1) = 1$

(ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$M_{11} =$ minor of element $a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 11$

$M_{12} =$ minor of element $a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6$

$M_{13} =$ minor of element $a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3$

$M_{21} =$ minor of element $a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = -4$

$M_{22} =$ minor of element $a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2$

$M_{23} =$ minor of element $a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{31} =$ minor of element $a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = -20$

$M_{32} =$ minor of element $a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -13$

$M_{33} =$ minor of element $a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (11) = 11$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (6) = -6$

$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 (3) = 3$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 (1) = -1$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 (-20) = -20$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-13) = 13$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 (5) = 5$

Question 3:

Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Solution:

The given determinant is $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$M_{21} =$ minor of element $a_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -7$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-7) = 7$

$M_{22} =$ minor of element $a_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (7) = 7$

$M_{23} =$ minor of element $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7$

$A_{23} = (-1)^{2+3} M_{21} = (-1)^5 (7) = -7$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

Therefore,

$$\begin{aligned} \Delta &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 2(7) + 0(7) + 1(-7) \\ &= 14 - 7 \\ &= 7 \end{aligned}$$

Question 4:

Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Solution:

The given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Therefore,

$$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$$

$$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$$

$$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$$

$$A_{13}=(-1)^{1+3}M_{13}=(-1)^4(z-y)=z-y$$
$$A_{23}=(-1)^{2+3}M_{23}=(-1)^5(z-x)=-(z-x)=x-z$$
$$A_{33}=(-1)^{3+3}M_{33}=(-1)^6(y-x)=y-x$$

We know that Δ is equal to the sum of the product of the elements of the third column with their corresponding cofactors.

Therefore,

$$\begin{aligned}\Delta &= a_{13}A_{13}+a_{23}A_{23}+a_{33}A_{33}\\ &= yz(z-y)+zx(x-z)+xy(y-x)\\ &= yz^2-y^2z+x^2z-xz^2+xy^2-x^2y\\ &= (x^2z-y^2z)+(yz^2-xz^2)+(xy^2-x^2y)\\ &= z(x^2-y^2)+z^2(y-x)+xy(y-x)\\ &= z(x-y)(x+y)+z^2(y-x)+xy(y-x)\\ &= (x-y)[zx+zy-z^2-xy]\\ &= (x-y)[z(x-z)+y(z-x)]\\ &= (x-y)(z-x)[-z+y]\\ &= (x-y)(y-z)(z-x)\end{aligned}$$

Hence,

$$\Delta=(x-y)(y-z)(z-x)$$

Question 5:

If $\Delta=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$ and A_{ij} is the cofactor of a_{ij}, then the value of Δ is given by:

A. $a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33}$
B. $a_{11}A_{11}+a_{12}A_{21}+a_{13}A_{31}$
C. $a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}$
D. $a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31}$

Solution:

We know that Δ is equal to the sum of the product of the elements of a column or row with their corresponding cofactors.

$$\Delta=a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31}$$

Thus, the correct option is D.

EXERCISE 4.5

Question 1:

Find the adjoint of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Then,

$$A_{11} = 4 \qquad A_{12} = -3$$
$$A_{21} = -2 \qquad A_{22} = 1$$

Therefore,

$$adjA = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

Question 2:

Find the adjoint of the matrix $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{pmatrix}$

Then,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 \qquad A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12 \qquad A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$$
$$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \qquad A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$
$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11 \qquad A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1 \qquad A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

Therefore,

$$adjA=\begin{pmatrix}3 & 1 & -11\\ -12 & 5 & -1\\ 6 & 2 & 5\end{pmatrix}$$

Question 3:

Verify $A(adjA)=(adjA)A=|A|I$ for

$$\begin{pmatrix}2 & 3\\ -4 & -6\end{pmatrix}$$

Solution:

Let $A=\begin{pmatrix}2 & 3\\ -4 & -6\end{pmatrix}$

Then,

$$|A|=-12-(-12)$$
$$=0$$

Also,

$$|A|I=0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Now,

$$A_{11}=-6 \qquad A_{12}=4$$
$$A_{21}=-3 \qquad A_{22}=2$$

Hence,

$$adjA=\begin{pmatrix}-6 & -3\\ 4 & 2\end{pmatrix}$$

Now,

$$A(adjA)=\begin{pmatrix}2 & 3\\ -4 & -6\end{pmatrix}\begin{pmatrix}-6 & -3\\ 4 & 2\end{pmatrix}$$
$$=\begin{pmatrix}-12+12 & -6+6\\ 24-24 & 12-12\end{pmatrix}$$
$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Also,

$$(adjA)A = \begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$$
$$= \begin{pmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $A(adjA) = (adjA)A = |A|I$.

Question 4:

Verify $A(adjA) = (adjA)A = |A|I$ for

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{pmatrix}$$

Solution:

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{pmatrix}$

Then,

$$|A| = 1(0-0)+1(9+2)+2(0-0)$$
$$= 11$$

Also,

$$|A|I = 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

Now,

$$A_{11} = 0 \qquad A_{12} = -11 \qquad A_{13} = 0$$
$$A_{21} = 3 \qquad A_{22} = 1 \qquad A_{23} = -1$$
$$A_{31} = 2 \qquad A_{32} = 8 \qquad A_{33} = 3$$

Hence,

$$adjA = \begin{pmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{pmatrix}$$

Now,

$$A(adjA) = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{pmatrix} = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

Also,

$$(adjA)A = \begin{pmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 2+0+9 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

Hence, $A(adjA) = (adjA)A = |A|I$.

Question 5:

Find the inverse of each of the matrix $\begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix}$ (if it exists).

Solution:

Let $A = \begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix}$

Then,

$$\begin{aligned}|A| &= 6+8 \\ &= 14\end{aligned}$$

Now,

$$\begin{aligned}A_{11} &= 3 \qquad & A_{12} &= -4 \\ A_{21} &= 2 \qquad & A_{22} &= 2\end{aligned}$$

Therefore,

$$adjA = \begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}$$

Hence,

$$\begin{aligned}A^{-1} &= \frac{1}{|A|} adjA \\ &= \frac{1}{14}\begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}\end{aligned}$$

Question 6:

Find the inverse of each of the matrix $\begin{pmatrix} -1 & 5 \\ -3 & 2 \end{pmatrix}$ (if it exists)

Solution:

Let $A = \begin{pmatrix} -1 & 5 \\ -3 & 2 \end{pmatrix}$

Then,

$$|A| = -2+15 = 13$$

Now,

$$\begin{aligned}A_{11} &= 2 \qquad & A_{12} &= 3 \\ A_{21} &= -5 \qquad & A_{22} &= -1\end{aligned}$$

Therefore,

$$adjA = \begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$$

Hence,

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= \frac{1}{13}\begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$$

Question 7:

Find the inverse of each of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$ (if it exists)

Solution:

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$

Then,

$$|A| = 1(10-0) - 2(0-0) + 3(0-0)$$
$$= 10$$

Now,

$$A_{11} = 10 \qquad A_{12} = 0 \qquad A_{13} = 0$$
$$A_{21} = -10 \qquad A_{22} = 5 \qquad A_{23} = 0$$
$$A_{31} = 2 \qquad A_{32} = -4 \qquad A_{33} = 2$$

Therefore,

$$adjA = \begin{pmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

Hence,

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= \frac{1}{10}\begin{pmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

Question 8:

Find the inverse of each of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$ (if it exists)

Solution:

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$

Then,

$$|A| = 1(-3-0)-0+0 = -3$$

Now,

$$\begin{matrix} A_{11} = -3 & A_{12} = 3 & A_{13} = -9 \\ A_{21} = 0 & A_{22} = -1 & A_{23} = -2 \\ A_{31} = 0 & A_{32} = 0 & A_{33} = 3 \end{matrix}$$

Therefore,

$$adjA = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{pmatrix}$$

Hence,

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{-1}{3}\begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{pmatrix}$$

Question 9:

Find the inverse of each of the matrix $\begin{pmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{pmatrix}$ (if it exists)

Solution:

Let $A=\begin{pmatrix}2&1&3\\4&-1&0\\-7&2&1\end{pmatrix}$

Then,

$$\begin{aligned}|A|&=2(-1-0)-1(4-0)+3(8-7)\\&=2(-1)-1(4)+3(1)\\&=-3\end{aligned}$$

Now,

$$\begin{matrix}A_{11}=-1 & A_{12}=-4 & A_{13}=1\\ A_{21}=5 & A_{22}=23 & A_{23}=-11\\ A_{31}=3 & A_{32}=12 & A_{33}=-6\end{matrix}$$

Therefore,

$$adjA=\begin{pmatrix}-1&5&3\\-4&23&12\\1&-11&-6\end{pmatrix}$$

Hence,

$$\begin{aligned}A^{-1}&=\frac{1}{|A|}adjA\\&=-\frac{1}{3}\begin{pmatrix}-1&5&3\\-4&23&12\\1&-11&-6\end{pmatrix}\end{aligned}$$

Question 10:

Find the inverse of each of the matrix $\begin{pmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{pmatrix}$ (if it exists)

Solution:

Let $A=\begin{pmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{pmatrix}$

Then, expanding along C_1,

$$|A| = 1(8-6) - 0 + 3(3-4) = 2-3$$
$$= -1$$

Now,

$$A_{11} = 2 \qquad A_{12} = -9 \qquad A_{13} = -6$$
$$A_{21} = 0 \qquad A_{22} = -2 \qquad A_{23} = -1$$
$$A_{31} = -1 \qquad A_{32} = 3 \qquad A_{33} = 2$$

Therefore,

$$adjA = \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}$$

Hence,

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= -1 \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$$

Question 11:

Find the inverse of each of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$ (if it exists)

Solution:

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$

Then,

$$|A| = 1(-\cos^2\alpha - \sin^2\alpha) = -(\cos^2\alpha + \sin^2\alpha)$$
$$= -1$$

Now,

$$A_{11} = -\cos^2\alpha - \sin^2\alpha = -1 \qquad A_{12} = 0 \qquad A_{13} = 0$$
$$A_{21} = 0 \qquad A_{22} = -\cos\alpha \qquad A_{23} = -\sin\alpha$$
$$A_{31} = 0 \qquad A_{32} = -\sin\alpha \qquad A_{33} = \cos\alpha$$

Therefore,

$$adjA = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

Hence,

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= -1\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$$

Question 12:

Let $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

Let $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$

Then,

$$|A| = 15 - 14$$
$$= 1$$

Now,

$$A_{11} = 5 \qquad A_{12} = -2$$
$$A_{21} = -7 \qquad A_{22} = 3$$

Then,

$$adjA = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$$

Now,

Let $B = \begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix}$

Then,

$$|B| = 54 - 56$$
$$= -2$$

Now,

$$A_{11} = 9 \qquad A_{12} = -7$$
$$A_{21} = -8 \qquad A_{22} = 6$$

Then,

$$adjB = \begin{pmatrix} 9 & -8 \\ -7 & 6 \end{pmatrix}$$

Therefore,

$$B^{-1} = \frac{1}{|B|} adjB$$
$$= -\frac{1}{2}\begin{pmatrix} 9 & -8 \\ -7 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{pmatrix}$$

Now,

$$B^{-1}A^{-1}=\begin{pmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{pmatrix}\begin{pmatrix}5 & -7\\ -2 & 3\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{45}{2}-8 & \frac{63}{2}+12\\ \frac{35}{2}+6 & -\frac{49}{2}-9\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & \frac{-67}{2}\end{pmatrix} \qquad \ldots(1)$$

Also,

$$AB=\begin{pmatrix}3 & 7\\ 2 & 5\end{pmatrix}\begin{pmatrix}6 & 8\\ 7 & 9\end{pmatrix}$$
$$=\begin{pmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{pmatrix}$$
$$=\begin{pmatrix}67 & 87\\ 47 & 61\end{pmatrix}$$

Then, we have

$$|AB|=67(61)-87(47)$$
$$=4087-4089$$
$$=-2$$

Therefore,

$$adj(AB)=\begin{pmatrix}61 & -87\\ -47 & 67\end{pmatrix}$$

Thus,

$$(AB)^{-1}=\frac{1}{|AB|}adj(AB)$$
$$=-\frac{1}{2}\begin{pmatrix}61 & -87\\ -47 & 67\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{pmatrix} \qquad \ldots(2)$$

From (1) and (2),

$$(AB)^{-1} = B^{-1}A^{-1}$$

Hence, proved.

Question 13:

If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 5A + 7I = 0$. Hence find A^{-1}.

Solution:

Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

Therefore,

$$\begin{aligned} A^2 = A.A &= \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix} \end{aligned}$$

Now,

$$\begin{aligned} A^2 - 5A + 7I &= \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix} - 5\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix} - \begin{pmatrix} 15 & 5 \\ -5 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Hence, $A^2 - 5A + 7I = 0$.

Now,

$$\Rightarrow A.A-5A=-7I$$

$$\Rightarrow A.A\left(A^{-1}\right)-5A.A^{-1}=-7IA^{-1} \qquad \left[\text{post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)-5I=-7A^{-1}$$

$$\Rightarrow AI-5I=-7A^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{7}(A-5I)$$

$$\Rightarrow A^{-1}=\frac{1}{7}(5I-A)$$

$$\Rightarrow A^{-1}=\frac{1}{7}\left[\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}-\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}\right]$$

$$\Rightarrow A^{-1}=\frac{1}{7}\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}$$

Thus,

$$A^{-1}=\frac{1}{7}\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}$$

Question 14:

For the matrix $A=\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}$, find the numbers a and b such that $A^2+aA+bI=0$.

Solution:

Let $A=\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}$

Therefore,

$$A^2=A.A=\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{pmatrix}=\begin{pmatrix}11 & 8\\ 4 & 3\end{pmatrix}$$

Now, $A^2+aA+bI=0$.

Hence,

$$\Rightarrow (A.A)A^{-1} + aA.A^{-1} + bIA^{-1} = 0 \qquad \left[\text{post-multiplying by } A^{-1} \text{ as } |A| \neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right) + aI + b\left(IA^{-1}\right) = 0$$

$$\Rightarrow AI + aI + bA^{-1} = 0$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{b}(A + aI) \qquad \ldots(1)$$

Now,

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{1}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \qquad \ldots(2)$$

From (1) and (2), we have,

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \frac{1}{b}\left[\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = -\frac{1}{b}\begin{pmatrix} 3+a & 2 \\ 1 & a \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{-3-a}{b} & -\frac{2}{b} \\ -\frac{1}{b} & \frac{-1-a}{b} \end{pmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$\Rightarrow -\frac{1}{b} = -1$$

$$\Rightarrow b = 1$$

Also,

$$\Rightarrow \frac{-3-a}{b}=1$$
$$\Rightarrow -3-a=1$$
$$\Rightarrow a=-4$$

Thus, $a=-4$ and $b=1$.

Question 15:

For the matrix $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}$, show that $A^3-6A^2+5A+11I=0$. Hence, find A^{-1}.

Solution:

Let $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}$

Therefore,

$$A^2=A.A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}$$
$$=\begin{pmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{pmatrix}=\begin{pmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{pmatrix}$$

And,

$$A^3=A^2.A=\begin{pmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}$$
$$=\begin{pmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{pmatrix}=\begin{pmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{pmatrix}$$

Hence,

$$A^3-6A^2+5A+11I=\begin{pmatrix}8&7&1\\-23&27&-69\\32&-13&58\end{pmatrix}-6\begin{pmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{pmatrix}+5\begin{pmatrix}1&1&1\\1&2&-3\\2&-1&3\end{pmatrix}+11\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}8&7&1\\-23&27&-69\\32&-13&58\end{pmatrix}-\begin{pmatrix}24&12&6\\-18&48&-84\\42&-18&84\end{pmatrix}+\begin{pmatrix}5&5&5\\5&10&-15\\10&-5&15\end{pmatrix}+\begin{pmatrix}11&0&0\\0&11&0\\0&0&11\end{pmatrix}$$

$$=\begin{pmatrix}24&12&6\\-18&48&-84\\42&-18&84\end{pmatrix}-\begin{pmatrix}24&12&6\\-18&48&-84\\42&-18&84\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$=0$$

Thus, $A^3-6A^2+5A+11I=0$

Now,

$$\Rightarrow A^3-6A^2+5A+11I=0$$

$$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+5AA^{-1}+11IA^{-1}=0 \qquad \left[\text{post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow AA\left(AA^{-1}\right)-6A\left(AA^{-1}\right)+5\left(AA^{-1}\right)=-11\left(IA^{-1}\right)$$

$$\Rightarrow A^2-6A+5I=-11A^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{11}\left(A^2-6A+5I\right) \qquad \ldots(1)$$

Now,

$$A^2-6A+5I=\begin{pmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{pmatrix}-6\begin{pmatrix}1&1&1\\1&2&-3\\2&-1&3\end{pmatrix}+5\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{pmatrix}-\begin{pmatrix}6&6&6\\6&12&-18\\12&-6&18\end{pmatrix}+\begin{pmatrix}5&0&0\\0&5&0\\0&0&5\end{pmatrix}$$

$$=\begin{pmatrix}9&2&1\\-3&13&-14\\7&-3&19\end{pmatrix}-\begin{pmatrix}6&6&6\\6&12&-18\\12&-6&18\end{pmatrix}$$

$$=\begin{pmatrix}3&-4&-5\\-9&1&4\\-5&3&1\end{pmatrix} \qquad \ldots(2)$$

From equation (1) and (2)

$$A^{-1}=-\frac{1}{11}\begin{pmatrix}3&-4&-5\\-9&1&4\\-5&3&1\end{pmatrix}$$

$$=\frac{1}{11}\begin{pmatrix}-3&4&5\\9&-1&-4\\5&-3&-1\end{pmatrix}$$

Question 16:

If $A=\begin{pmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix}$, verify that $A^3-6A^2+9A-4I=0$. Hence, find A^{-1}.

Solution:

Let $A=\begin{pmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix}$

Therefore,

$$A^2 = A.A$$

$$= \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

And

$$A^3 = A^2.A$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

Now,

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - \begin{pmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{pmatrix} + \begin{pmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{pmatrix} - \begin{pmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

Thus,

$$A^3 - 6A^2 + 9A - 4I = 0$$

Now,

$$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0$$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 9AA^{-1} - 4IA^{-1} = 0 \qquad \left[\text{post-multiplying by } A^{-1} \text{ as } |A| \neq 0\right]$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 9(AA^{-1}) = 4(IA^{-1})$$

$$\Rightarrow AAI - 6AI + 9I = 4A^{-1}$$

$$\Rightarrow A^2 - 6A + 9I = 4A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) \qquad \ldots(1)$$

Now,

$$A^2 - 6A + 9I = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} \qquad \ldots(2)$$

From equations (1) and (2),

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

Question 17:

Let A be a non-singular square matrix of order 3×3. Then $|adjA|$ is equal to:

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

Solution:

Since A be a non-singular square matrix of order 3×3

$$(adjA)A = |A|I$$
$$= \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

Therefore,

$$|(adjA)A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$
$$|adjA||A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= |A|^3 I$$
$$|adjA| = |A|^2$$

Thus, the correct option is B.

Question 18:

If A is an invertible matrix of order 2, the $\det(A^{-1})$ is equal to:

(A) $\det(A)$ (B) $\dfrac{1}{\det(A)}$ (C) 1 (D) 0

Solution:

Since A is an invertible matrix, A^{-1} exists and $A^{-1} = \dfrac{1}{|A|} adjA$.

As matrix A is of order 2, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then,

$$|A| = ad - bc$$

And

$$adjA = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Now,

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \begin{pmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{pmatrix}$$

Hence,

$$|A^{-1}| = \begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix}$$

$$|A^{-1}| = \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

$$= \frac{1}{|A|^2}(ad - bc)$$

$$= \frac{1}{|A|^2} . |A|$$

$$= \frac{1}{|A|}$$

Hence,

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

Thus, the correct option is B.

EXERCISE 4.6

Question 1:

Examine the consistency of the system of equations:

$x+2y=2$

$2x+3y=3$

Solution:

The given system of equations is: $\begin{aligned} x+2y=2 \\ 2x+3y=3 \end{aligned}$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Hence,

$$\begin{aligned} |A| &= 1(3)-2(2) \\ &= 3-4 \\ &= -1 \\ &\neq 0 \end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Thus, the given system of equations is consistent.

Question 2:

Examine the consistency of the system of equations:

$2x-y=5$

$x+y=4$

Solution:

The given system of equations is: $\begin{aligned} 2x-y=5 \\ x+y=4 \end{aligned}$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Hence,

$$\begin{aligned}|A| &= 2(1)-1(-1)\\ &= 2+1\\ &= 3\\ &\neq 0\end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 3:

Examine the consistency of the system of equations:

$x+3y=5$

$2x+6y=8$

Solution:

The given system of equations is: $\begin{aligned}x+3y&=5\\2x+6y&=8\end{aligned}$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix}1 & 3\\ 2 & 6\end{pmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 8\end{bmatrix}$

Hence,

$$\begin{aligned}|A| &= 1(6)-3(2)\\ &= 6-6\\ &= 0\end{aligned}$$

So, A is a singular matrix.

Now,

$$(adjA)=\begin{pmatrix}6 & -3\\ -2 & 1\end{pmatrix}$$

Therefore,

$$
\begin{aligned}
(adjA)B &= \begin{pmatrix} 6 & -5 \\ -2 & 1 \end{pmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} \\
&= \begin{pmatrix} 30-24 \\ -10+8 \end{pmatrix} \\
&= \begin{bmatrix} 6 \\ -2 \end{bmatrix} \\
&\neq 0
\end{aligned}
$$

Thus, the solution of the given system of equations does not exist.

Hence, the system of equations is inconsistent.

Question 4:

Examine the consistency of the system of equations:

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Solution:

The given system of equations is:

$$
\begin{aligned}
x + y + z &= 1 \\
2x + 3y + 2z &= 2 \\
ax + ay + 2az &= 4
\end{aligned}
$$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Hence,

$$
\begin{aligned}
|A| &= 1(6a-2a) - 1(4a-2a) + 1(2a-3a) \\
&= 4a - 2a - a \\
&= 4a - 3a \\
&= a \neq 0
\end{aligned}
$$

So, A is non-singular.

Therefore, A^{-1} exists.

Thus, the given system of equations is consistent.

Question 5:

Examine the consistency of the system of equations:

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Solution:

The given system of equations is:
$$3x - y - 2z = 2$$
$$2y - z = -1$$
$$3x - 5y = 3$$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{pmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Hence,

$$|A| = 3(0-5) - 0 + 3(1+4)$$
$$= -15 + 15$$
$$= 0$$

So, A is a singular matrix.

Now,

$$(adjA) = \begin{pmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{pmatrix}$$

Therefore,

$$(adjA)B=\begin{pmatrix}-5 & 10 & 5\\ -3 & 6 & 3\\ -6 & 12 & 6\end{pmatrix}\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}$$
$$=\begin{bmatrix}-10-10+15\\ -6-6+9\\ -12-12+18\end{bmatrix}$$
$$=\begin{bmatrix}-5\\ -3\\ -6\end{bmatrix}$$
$$\neq 0$$

Thus, the solution of the given system of equations does not exist.

Hence, the system of equations is inconsistent.

Question 6:

Examine the consistency of the system of equations:

$5x-y+4z=5$

$2x+3y+5z=2$

$5x-2y+6z=-1$

Solution:

The given system of equations is:
$$\begin{aligned}5x-y+4z&=5\\ 2x+3y+5z&=2\\ 5x-2y+6z&=-1\end{aligned}$$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix}5 & -1 & 4\\ 2 & 3 & 5\\ 5 & -2 & 6\end{pmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 2\\ -1\end{bmatrix}$

Hence,
$$\begin{aligned}|A|&=5(18+10)+1(12-25)+4(-4-15)\\ &=5(28)+1(-13)+4(-19)\\ &=140-13-76\\ &=51\neq 0\end{aligned}$$

So, A is nonsingular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 7:

Solve system of linear equations, using matrix method.

$5x+2y=4$

$7x+3y=5$

Solution:

The given system of equations is: $\begin{matrix} 5x+2y=4 \\ 7x+3y=5 \end{matrix}$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Hence,

$$\begin{aligned} |A| &= 15-14 \\ &= 1 \\ &\neq 0 \end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$\begin{aligned} A^{-1} &= \frac{1}{|A|}(adjA) \\ &= \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} \end{aligned}$$

Then,

$$\Rightarrow X = A^{-1}B$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence, $x = 2$ and $y = -3$

Question 8:

Solve system of linear equations, using matrix method.

$2x - y = -2$

$3x + 4y = 3$

Solution:

The given system of equations is: $\begin{matrix} 2x - y = -2 \\ 3x + 4y = 3 \end{matrix}$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Hence,

$$|A| = 8 + 3$$
$$= 11$$
$$\neq 0$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$A^{-1} = \frac{1}{|A|}(adjA)$$
$$= \frac{1}{11}\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$$

Therefore,

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Hence, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$

Question 9:

Solve system of linear equations, using matrix method.

$4x - 3y = 3$

$3x - 5y = 7$

Solution:

The given system of equations is: $\begin{array}{l} 4x - 3y = 3 \\ 3x - 5y = 7 \end{array}$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{pmatrix} 4 & -3 \\ 3 & -5 \end{pmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Hence,

$$\begin{aligned} |A| &= -20 + 9 \\ &= -11 \\ &\neq 0 \end{aligned}$$

So, A is nonsingular.

Therefore, A^{-1} exists.

Now,

$$A^{-1} = \frac{1}{|A|}(adjA)$$
$$= -\frac{1}{11}\begin{pmatrix} -5 & 3 \\ -3 & 4 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 3 & -4 \end{pmatrix}$$

Therefore,

$$\Rightarrow X = A^{-1}B$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 3 & -4 \end{pmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 3 & -4 \end{pmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 15-21 \\ 9-28 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -6 \\ -19 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-6}{11} \\ \frac{-19}{11} \end{bmatrix}$$

Hence, $x = \frac{-6}{11}$ and $y = \frac{-19}{11}$

Question 10:

Solve system of linear equations, using matrix method.

$5x + 2y = 3$

$3x + 2y = 5$

Solution:

The given system of equations is: $\begin{aligned} 5x + 2y = 3 \\ 3x + 2y = 5 \end{aligned}$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Hence,

$$\begin{aligned} |A| &= 10 - 6 \\ &= 4 \\ &\neq 0 \end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$\begin{aligned} A^{-1} &= \frac{1}{|A|}(adjA) \\ &= \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} &\Rightarrow X = A^{-1}B \\ &\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \end{aligned}$$

Hence, $x = -1$ and $y = 4$

Question 11:

Solve system of linear equations, using matrix method.

$2x+y+z=1$

$x-2y-z=\frac{3}{2}$

$3y-5z=9$

Solution:

The given system of equations is:
$$2x+y+z=1$$
$$x-2y-z=\frac{3}{2}$$
$$3y-5z=9$$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix}2 & 1 & 1\\ 1 & -2 & -1\\ 0 & 3 & -5\end{pmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}$

Hence,

$$\begin{aligned}|A| &= 2(10+3)-1(-5-3)+0\\ &= 2(13)-1(-8)\\ &= 26+8\\ &= 34\\ &\neq 0\end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$\begin{array}{lll} A_{11}=13 & A_{12}=5 & A_{13}=3\\ A_{21}=8 & A_{22}=-10 & A_{23}=-6\\ a_{31}=1 & A_{32}=3 & A_{33}=-5\end{array}$$

Hence,

$$\begin{aligned}A^{-1} &= \frac{1}{|A|}(adjA)\\ &= \frac{1}{34}\begin{pmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -6 & -5\end{pmatrix}\end{aligned}$$

Therefore,

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Hence, $x = 1, y = \frac{1}{2}$ and $z = \frac{-3}{2}$

Question 12:

Solve system of linear equations, using matrix method.

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Solution:

The given system of equations is:
$$\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}$$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Hence,

$$\begin{aligned}|A| &= 1(1+3)+1(2+3)+1(2-1)\\ &= 4+5+1\\ &= 10\\ &\neq 0\end{aligned}$$

So, A is nonsingular.

Therefore, A^{-1} exists.

Now,

$$\begin{array}{lll} A_{11}=4 & A_{12}=-5 & A_{13}=1\\ A_{21}=2 & A_{22}=0 & A_{23}=-2\\ a_{31}=2 & A_{32}=5 & A_{33}=3\end{array}$$

Hence,

$$\begin{aligned}A^{-1} &= \frac{1}{|A|}(adjA)\\ &= \frac{1}{10}\begin{pmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{pmatrix}\end{aligned}$$

Therefore,

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 2, y = -1$ and $z = 1$

Question 13:

Solve system of linear equations, using matrix method.

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Solution:

The given system of equations is:
$$\begin{aligned} 2x + 3y + 3z &= 5 \\ x - 2y + z &= -4 \\ 3x - y - 2z &= 3 \end{aligned}$$

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Hence,

$$
\begin{aligned}
|A| &= 2(4+1)-3(-2-3)+3(-1+6) \\
&= 10+15+15 \\
&= 40 \\
&\neq 0
\end{aligned}
$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$
\begin{array}{lll}
A_{11}=5 & A_{12}=5 & A_{13}=5 \\
A_{21}=3 & A_{22}=-13 & A_{23}=11 \\
A_{31}=9 & A_{32}=1 & A_{33}=-7
\end{array}
$$

Hence,

$$
\begin{aligned}
A^{-1} &= \frac{1}{|A|}(adjA) \\
&= \frac{1}{40}\begin{pmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{pmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Rightarrow X = A^{-1}B \\
&\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{pmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{pmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{pmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{pmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}
\end{aligned}
$$

Hence, $x=1, y=2$ and $z=-1$

Question 14:

Solve system of linear equations, using matrix method.

$$x-y+2z=7$$
$$3x+4y-5z=-5$$
$$2x-y+3z=12$$

Solution:

The given system of equations is:
$$\begin{aligned} x-y+2z&=7\\ 3x+4y-5z&=-5\\ 2x-y+3z&=12 \end{aligned}$$

The given system of equations can be written in the form of $AX=B$, where

$$A=\begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{pmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B=\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

Hence,

$$\begin{aligned} |A| &= 1(12-5)+1(9+10)+2(-3-8) \\ &= 7+19-22 \\ &= 4 \\ &\neq 0 \end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$\begin{array}{lll} A_{11}=7 & A_{12}=-19 & A_{13}=-11 \\ A_{21}=1 & A_{22}=-1 & A_{23}=-1 \\ a_{31}=-3 & A_{32}=11 & A_{33}=7 \end{array}$$

Hence,

$$A^{-1} = \frac{1}{|A|}(adjA)$$
$$= \frac{1}{4}\begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix}$$

Therefore,

$$\Rightarrow X = A^{-1}B$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2, y = 1$ and $z = 3$

Question 15:

If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Solution:

It is given that $A=\begin{pmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{pmatrix}$

Therefore,

$$\begin{aligned}|A| &= 2(-4+4)+3(-6+4)+5(3-2)\\ &= 0-6+5\\ &= -1\\ &\neq 0\end{aligned}$$

Now,

$$\begin{array}{lll} A_{11}=0 & A_{12}=2 & A_{13}=1\\ A_{21}=-1 & A_{22}=-9 & A_{23}=-5\\ a_{31}=2 & A_{32}=23 & A_{33}=13 \end{array}$$

Hence,

$$\begin{aligned}A^{-1} &= \frac{1}{|A|}(adjA)\\ &= -\begin{pmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{pmatrix}=\begin{pmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{pmatrix}\end{aligned}$$

The given system of equations can be written in the form of $AX=B$, where

$A=\begin{pmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{pmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$

The solution of the system of equations is given by $X=A^{-1}B$.

Therefore,

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2$ and $z = 3$

Question 16:

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

Solution:

Let the cost of onions, wheat, and rice per kg in ₹ be x, y and z respectively.

Then, the given situation can be represented by a system of equations as:

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Therefore,

$$\begin{aligned} |A| &= 4(12-12) - 3(6-36) + 2(4-24) \\ &= 0 + 90 - 40 \\ &= 50 \\ &\neq 0 \end{aligned}$$

So, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$\begin{array}{lll} A_{11} = 0 & A_{12} = 30 & A_{13} = -20 \\ A_{21} = -5 & A_{22} = 0 & A_{23} = 10 \\ A_{31} = 10 & A_{32} = -20 & A_{33} = 10 \end{array}$$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{1}{|A|}(adjA) \\ &= \frac{1}{50}\begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix} \end{aligned}$$

Hence,

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Thus, $x = 5, y = 8$ and $z = 8$

Hence, the cost of onions is ₹ 5 per kg the cost of wheat is ₹ 8 per kg, and the cost of rice is ₹ 8 per kg.

MISCELLANEOUS EXERCISE

Question 1:

Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Solution:

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$= x\left(-x^2-1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$= -x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$$

$$= -x^3 - x + x$$

$$= -x^3$$

Hence, Δ is independent of θ.

Question 2:

Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Solution:

$$LHS = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \qquad [R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3]$$

$$= \frac{1}{abc}\cdot abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad [\text{Taking out factor } abc \text{ from } C_3]$$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \qquad [C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3]$$

$$= RHS$$

Hence, proved.

Question 3:

Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

Solution:

Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along C_3,

$$\Delta = -\sin\alpha\left(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha\right) + \cos\alpha\left(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta\right)$$

$$= \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right) + \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right)$$

$$= \sin^2\alpha(1) + \cos^2\alpha(1)$$

$$= 1$$

Question 4:

If a, b, c are real numbers and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, show that either $a+b+c=0$ or $a=b=c$.

Solution:

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix} \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

Expanding R_1,

$$\begin{aligned}\Delta &= 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right] \\ &= 2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right] \\ &= 2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]\end{aligned}$$

It is given that $\Delta = 0$.

Hence,

$$2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]=0$$

Either $(a+b+c)=0$ or $\left[ab+bc+ca-a^2-b^2-c^2\right]=0$

Now,

$\Rightarrow ab+bc+ca-a^2-b^2-c^2=0$
$\Rightarrow -2ab-2ac-2ca+2a^2+2b^2+2c^2=0$
$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$
$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0$ $\left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$
$\Rightarrow (a-b)=(b-c)=(c-a)=0$
$\Rightarrow a=b=c$

Hence, if $\Delta=0$, then either $(a+b+c)=0$ or $a=b=c$.

Question 5:

Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a\neq 0$.

Solution:

$$\Rightarrow \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0 \qquad [R_1 \to R_1+R_2+R_3]$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix}=0 \qquad [C_2 \to C_2-C_1 \text{ and } C_3 \to C_3-C_1]$$

Expanding along R_1,
$\Rightarrow (3x+a)\left[1\times a^2\right]=0$
$\Rightarrow a^2(3x+a)=0$

Since $a\neq 0$

Therefore,

$$\Rightarrow 3x + a = 0$$

$$\Rightarrow x = -\frac{a}{3}$$

Question 6:

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

Solution:

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix} \quad [\text{Taking out common factors } a, b \text{ and } c \text{ from } C_1, C_2 \text{ and } C_3]$$

$$= abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix} \quad [R_2 \to R_2 + R_1]$$

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix} \quad [R_3 \to R_3 + R_2]$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

Expanding along R_3,

$$\begin{aligned}\Delta &= 2ab^2c\left[a(c-a)+a(a+c)\right]\\ &= 2ab^2c\left[ac-a^2+a^2+ac\right]\\ &= 2ab^2c(2ac)\\ &= 4a^2b^2c^2\end{aligned}$$

Hence, proved.

Question 7:

If $A^{-1}=\begin{vmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{vmatrix}$ and $B=\begin{vmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{vmatrix}$, find $(AB)^{-1}$.

Solution:

We know that $(AB)^{-1}=B^{-1}A^{-1}$.

It is given that $B=\begin{vmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{vmatrix}$

Therefore,

$$\begin{aligned}|B| &= 1(3)-2(-1)-2(-2)\\ &= 3+2-4\\ &= 5-4\\ &= 1\end{aligned}$$

Now,

$$\begin{array}{lll} B_{11}=3 & B_{12}=1 & B_{13}=2\\ B_{21}=2 & B_{22}=1 & B_{23}=2\\ B_{31}=6 & B_{32}=2 & B_{33}=5\end{array}$$

Hence,

$$adjB=\begin{pmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{pmatrix}$$

Now,

$$\begin{aligned}B^{-1} &= \frac{1}{|B|}adjB\\ &= \begin{pmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{pmatrix}\end{aligned}$$

Therefore,

$$\begin{aligned}(AB)^{-1} &= B^{-1}A^{-1}\\ &= \begin{pmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{pmatrix}\begin{pmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{pmatrix}\\ &= \begin{pmatrix}9-30+30 & -3+12-12 & 3-10+12\\ 3-15+10 & -1+6-4 & 1-5+4\\ 6-30+25 & -2+12-10 & 2-10+10\end{pmatrix}\\ &= \begin{pmatrix}9 & -3 & 5\\ -2 & 1 & 0\\ 1 & 0 & 2\end{pmatrix}\end{aligned}$$

Thus, $(AB)^{-1} = \begin{pmatrix}9 & -3 & 5\\ -2 & 1 & 0\\ 1 & 0 & 2\end{pmatrix}$.

Question 8:

Let $A = \begin{pmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{pmatrix}$ verify that

(i) $[adjA]^{-1} = adj(A)^{-1}$

(ii) $(A^{-1})^{-1} = A$

Solution:

It is given that $A = \begin{pmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{pmatrix}$

Therefore,

$$\begin{aligned}|A| &= 1(15-1)+2(-10-1)+1(-2-3)\\ &= 14-22-5\\ &= -13\end{aligned}$$

Now,

$$A_{11} = 14 \qquad A_{12} = 11 \qquad A_{13} = -5$$
$$A_{21} = 11 \qquad A_{22} = 4 \qquad A_{23} = -3$$
$$A_{31} = -5 \qquad A_{32} = -3 \qquad A_{33} = -1$$

Hence,

$$adjA = \begin{pmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{pmatrix}$$

Now,

$$\begin{aligned} A^{-1} &= \frac{1}{|A|}(adjA) \\ &= -\frac{1}{13}\begin{pmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{pmatrix} \\ &= \frac{1}{13}\begin{pmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{pmatrix} \end{aligned}$$

(i)

$$\begin{aligned} |adjA| &= 14(-4-9) - 11(-11-15) - 5(-33+20) \\ &= 14(-13) - 11(-26) - 5(-13) \\ &= -182 + 286 + 65 \\ &= 169 \end{aligned}$$

We have,

$$adj(adjA) = \begin{pmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{pmatrix}$$

Therefore,

$$[adjA]^{-1} = \frac{1}{|adjA|}\left(adj\left(adjA\right)\right)$$

$$= \frac{1}{169}\begin{pmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{pmatrix}$$

Now,

$$A^{-1} = -\frac{1}{13}\begin{pmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{pmatrix} = \begin{pmatrix} \frac{-14}{13} & \frac{-11}{13} & \frac{5}{13} \\ \frac{-11}{13} & \frac{-4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{pmatrix}$$

Therefore,

$$adj(A)^{-1} = \begin{pmatrix} \frac{-13}{169} & \frac{26}{169} & \frac{-13}{169} \\ \frac{26}{169} & \frac{-39}{169} & \frac{-13}{169} \\ \frac{-13}{169} & \frac{-13}{169} & \frac{-65}{169} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{pmatrix}$$

Hence, $[adjA]^{-1} = adj(A)^{-1}$ proved.

(ii)

$$A^{-1} = \frac{1}{13}\begin{pmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{pmatrix}$$

Hence,

$$adj(A)^{-1} = \begin{pmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{pmatrix}$$

Now,

$$|A^{-1}| = \left(\frac{1}{13}\right)^3 \left[-14(-4-9)+11(-11-26)+5(-33+20)\right]$$

$$= \left(\frac{1}{13}\right)^3 [-169]$$

$$= -\frac{1}{13}$$

Therefore,

$$\left(A^{-1}\right)^{-1}=\frac{adjA^{-1}}{|A|}=\frac{1}{\left(-\frac{1}{13}\right)}\times\begin{pmatrix}\frac{-1}{13} & \frac{2}{13} & \frac{-1}{13}\\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13}\\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13}\end{pmatrix}$$

$$=\begin{pmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{pmatrix}=A$$

Hence, $\left(A^{-1}\right)^{-1}=A$ proved.

Question 9:

Evaluate $\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$.

Solution:

$$\Delta=\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$$

$$=\begin{vmatrix}2(x+y) & 2(x+y) & 2(x+y)\\ y & x+y & x\\ x+y & x & y\end{vmatrix} \qquad [R_1\to R_1+R_2+R_3]$$

$$=2(x+y)\begin{vmatrix}1 & 1 & 1\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$$

$$=2(x+y)\begin{vmatrix}1 & 0 & 0\\ y & x & x-y\\ x+y & -y & -x\end{vmatrix} \qquad [C_2\to C_2-C_1 \text{ and } C_3\to C_3-C_1]$$

$$=2(x+y)\left[-x^2+y(x-y)\right] \qquad [\text{Expanding along } R_1]$$

$$=-2(x+y)\left(x^2+y^2-yx\right)$$

$$=-2\left(x^3+y^3\right)$$

Question 10:

Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$.

Solution:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= 1(xy - 0) \qquad [\text{Expanding along } C_1]$$

$$= xy$$

Question 11:

Using properties of determinants prove that:

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Solution:

$$\Delta=\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta & \beta^2 & \gamma+\alpha\\ \gamma & \gamma^2 & \alpha+\beta\end{vmatrix}$$

$$=\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta\\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma\end{vmatrix} \qquad [R_2\to R_2-R_1 \text{ and } R_3\to R_3-R_1]$$

$$=(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ 1 & \beta+\alpha & -1\\ 1 & \gamma+\alpha & -1\end{vmatrix}$$

$$=(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ 1 & \beta+\alpha & -1\\ 0 & \gamma-\beta & 0\end{vmatrix} \qquad [R_3\to R_3-R_2]$$

$$=(\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right] \qquad [\text{Expanding along } R_3]$$

$$=(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)$$

$$=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence, proved.

Question 12:

Using properties of determinants prove that:

$$\begin{vmatrix}x & x^2 & 1+px^3\\ y & y^2 & 1+py^3\\ z & z^2 & 1+pz^3\end{vmatrix}=(1+pxyz)(x-y)(y-z)(z-x)$$

Solution:

$$\Delta=\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$\Delta=\begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p\left(y^3-x^3\right) \\ z-x & z^2-x^2 & p\left(z^3-x^3\right) \end{vmatrix} \qquad \left[R_2\to R_2-R_1 \text{ and } R_3\to R_3-R_1\right]$$

$$\Delta=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 1 & z+x & p\left(z^2+x^2+xz\right) \end{vmatrix}$$

$$\Delta=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix} \qquad \left[R_3\to R_3-R_2\right]$$

$$\Delta=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

$$\Delta=(x-y)(z-y)(z-x)\left[(-1)(p)\left(xy^2+x^3+x^2y\right)+1+px^3+p(x+y+z)(xy)\right] \qquad \left[\text{Expanding along } R_3\right]$$

$$=(x-y)(y-z)(z-x)\left[-pxy^2-px^3-px^2y+1+px^3+px^2y+pxy^2+pxyz\right]$$

$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence, proved.

Question 13:

Using properties of determinants prove that:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}=3(a+b+c)(ab+bc+ca)$$

Solution:

$$\Delta=\begin{vmatrix}3a & -a+b & -a+c\\ -b+a & 3b & -b+c\\ -c+a & -c+b & 3c\end{vmatrix}$$

$$=\begin{vmatrix}a+b+c & -a+b & -a+c\\ a+b+c & 3b & -b+c\\ a+b+c & -c+b & 3c\end{vmatrix} \qquad [C_1 \to C_1+C_2+C_3]$$

$$=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\ 1 & 3b & -b+c\\ 1 & -c+b & 3c\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\ 0 & 2b+a & a-b\\ 0 & a-c & 2c+a\end{vmatrix} \qquad [R_2 \to R_2-R_1 \text{ and } R_3 \to R_3-R_1]$$

$$=(a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right] \qquad [\text{Expanding along } C_1]$$

$$=(a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$

$$=(a+b+c)(3ab+3bc+3ac)$$

$$=3(a+b+c)(ab+bc+ca)$$

Hence, proved.

Question 14:

Using properties of determinants prove that:

$$\begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 4+3p+2q\\ 3 & 6+3p & 10+6p+3q\end{vmatrix}=1$$

Solution:

$$\Delta=\begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 4+3p+2q\\ 3 & 6+3p & 10+6p+3q\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1+p & 1+p+q\\ 0 & 1 & 2+p\\ 0 & 3 & 7+3p\end{vmatrix}\qquad [R_2\to R_2-2R_1 \text{ and } R_3\to R_3-3R_1]$$

$$=\begin{vmatrix}1 & 1+p & 1+p+q\\ 0 & 1 & 2+p\\ 0 & 0 & 1\end{vmatrix}\qquad [R_3\to R_3-3R_2]$$

$$=1\begin{vmatrix}1 & 2+p\\ 0 & 1\end{vmatrix}\qquad [\text{Expanding along } C_1]$$

$$=1(1-0)=1$$

Hence, proved.

Question 15:

Using properties of determinants prove that:

$$\begin{vmatrix}\sin\alpha & \cos\alpha & \cos(\alpha+\delta)\\ \sin\beta & \cos\beta & \cos(\beta+\delta)\\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta)\end{vmatrix}=0$$

Solution:

$$\Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & \cos(\alpha+\delta)\\ \sin\beta & \cos\beta & \cos(\beta+\delta)\\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta)\end{vmatrix}$$

$$=\frac{1}{\sin\delta\cos\delta}\begin{vmatrix}\sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta-\sin\alpha\sin\delta\\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta-\sin\beta\sin\delta\\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta-\sin\gamma\sin\delta\end{vmatrix}$$

$$=\frac{1}{\sin\delta\cos\delta}\begin{vmatrix}\cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta-\sin\alpha\sin\delta\\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta-\sin\beta\sin\delta\\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta-\sin\gamma\sin\delta\end{vmatrix}\qquad [C_1\to C_1+C_3]$$

Here, two columns C_1 and C_2 are identical.

Therefore, $\Delta=0$

Hence, proved.

Question 16:
Solve the system of the following equations:

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$
$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Solution:

Let $\frac{1}{x}=p, \frac{1}{y}=q$ and $\frac{1}{z}=r$.

Then the given system of equations is as follows:

$$2p+3q+10r=4$$
$$4p-6q+5r=1$$
$$6p+9q-20r=2$$

This system can be written in the form of $AX=B$, where

$$A=\begin{pmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{pmatrix}, X=\begin{bmatrix}p\\ q\\ r\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

Therefore,

$$\begin{aligned}|A| &= 2(120-45)-3(-80-30)+10(36+36)\\ &=150+330+720\\ &=1200\end{aligned}$$

Thus, A is non-singular.

Therefore, A^{-1} exists.

Now,

$$\begin{array}{lll}A_{11}=75 & A_{12}=110 & A_{13}=72\\ A_{21}=150 & A_{22}=-100 & A_{23}=0\\ A_{31}=75 & A_{32}=30 & A_{33}=-24\end{array}$$

Hence,

$$A^{-1}=\frac{1}{|A|}(adjA)$$
$$=\frac{1}{1200}\begin{pmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{pmatrix}$$

Now,

$$\Rightarrow X=A^{-1}B$$
$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{pmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{pmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{3}\\ \frac{1}{5}\end{bmatrix}$$

Therefore,
$p=\frac{1}{2}, q=\frac{1}{3}$ and $r=\frac{1}{5}$

Hence, $x=2, y=3$ and $z=5$.

Question 17:

If a,b,c are in A.P, then the determinant $\begin{vmatrix}x+2 & x+3 & x+2a\\ x+3 & x+4 & x+2b\\ x+4 & x+5 & x+2c\end{vmatrix}$ is

(A) 0 (B) 1 (C) x (D) $2x$

Solution:

$$\Delta=\begin{vmatrix}x+2 & x+3 & x+2a\\ x+3 & x+4 & x+2b\\ x+4 & x+5 & x+2c\end{vmatrix}$$

$$=\begin{vmatrix}x+2 & x+3 & x+2a\\ x+3 & x+4 & x+(a+c)\\ x+4 & x+5 & x+2c\end{vmatrix} \qquad (2b=a+c \text{ as } a,b,c \text{ are in A.P})$$

$$=\begin{vmatrix}-1 & -1 & a-c\\ x+3 & x+4 & x+(a+c)\\ 1 & 1 & c-a\end{vmatrix} \qquad [R_1\to R_1-R_2 \text{ and } R_3\to R_3-R_2]$$

$$=\begin{vmatrix}0 & 0 & 0\\ x+3 & x+4 & x+a+c\\ 1 & 1 & c-a\end{vmatrix} \qquad [R_1\to R_1+R_3]$$

Here, all the elements of the first row are zero.

Hence, we have $\Delta=0$

Thus, the correct option is A.

Question 18:

If x,y,z are non-zero real numbers, then the inverse of matrix $A=\begin{pmatrix}x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}$ is

(A) $\begin{pmatrix}x^{-1} & 0 & 0\\ 0 & y^{-1} & 0\\ 0 & 0 & z^{-1}\end{pmatrix}$

(B) $xyz\begin{pmatrix}x^{-1} & 0 & 0\\ 0 & y^{-1} & 0\\ 0 & 0 & z^{-1}\end{pmatrix}$

(C) $\frac{1}{xyz}\begin{pmatrix}x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}$

(D) $\frac{1}{xyz}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Solution:

It is given that $A=\begin{pmatrix}x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}$

Hence,

$$\begin{aligned}|A| &= x(yz-0)\\ &= xyz\\ &\neq 0\end{aligned}$$

Now,

$$\begin{array}{lll} A_{11} = yz & A_{12} = 0 & A_{13} = 0 \\ A_{21} = 0 & A_{22} = xz & A_{23} = 0 \\ A_{31} = 0 & A_{32} = 0 & A_{33} = xy \end{array}$$

Therefore,

$$\begin{aligned}A^{-1} &= \frac{1}{|A|}(adjA)\\ &= \frac{1}{xyz}\begin{pmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{pmatrix}\\ &= \begin{pmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{pmatrix}\\ &= \begin{pmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}\\ &= \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}\end{aligned}$$

Thus, the correct option is A.

Question 19:

Let $A = \begin{pmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{pmatrix}$, where $0 \le \theta \le 2\pi$, then:

(A) $Det(A) = 0$ (B) $Det(A) \in (2, \infty)$

(C) $Det(A) \in (2,4)$ (D) $Det(A) \in [2,4]$

Solution:

It is given that $A = \begin{pmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{pmatrix}$

Hence,

$$\begin{aligned} |A| &= 1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1) \\ &= 1 + \sin^2\theta + \sin^2\theta + 1 \\ &= 2 + 2\sin^2\theta \\ &= 2(1+\sin^2\theta) \end{aligned}$$

Now,

$$\begin{aligned} &\Rightarrow 0 \le \theta \le 2\pi \\ &\Rightarrow -1 \le \sin\theta \le 1 \\ &\Rightarrow 0 \le \sin^2\theta \le 1 \\ &\Rightarrow 1 \le 1+\sin^2\theta \le 2 \\ &\Rightarrow 2 \le 2(1+\sin^2\theta) \le 4 \end{aligned}$$

Therefore,

$$Det(A) \in [2,4]$$

Thus, the correct option is D.